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The general common Hermitian nonnegative-definite solution to the matrix equations $AXA^* = BB^*$ and $CXC^* = DD^*$ with applications in statistics

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Abstract

We deduce a necessary and sufficient condition for the matrix equations $AXA^* = BB^*$ and $CXC^* = DD^*$ to have a common Hermitian nonnegative-definite solution, and a representation of the general common Hermitian nonnegative-definite solution to these two equations when they have such common solutions. Thereby, we solve a statistical problem which is concerned in testing linear hypotheses about regression coefficients in the multivariate linear model. This paper is a revision of Young et al. (J. Multivariate Anal. 68 (1999) 165) whose mistake was pointed out in (Linear Algebra Appl. 321 (2000) 123).

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1. Introduction

Let $\mathbb{R}^{m\times n}$ (respectively, $\mathbb{C}^{m\times n}$) be the set of all $m\times n$ real (respectively, complex) matrices. We denote by $\mathbb{O}_n$, $\mathbb{R}_n^{\geqslant}$ and $\mathbb{R}_n^{>}$ the subsets of $\mathbb{R}^{n\times n}$ consisting of all orthogonal matrices, symmetric nonnegative-definite matrices and symmetric positive-definite matrices, respectively. For a matrix X, let X', X^*, X^+, $\mathcal{N}(X)$ and $\mathcal{R}(X)$ be the transpose, the conjugate transpose, the Moore–Penrose generalized inverse, the kernel space and the column space of X, respectively. We denote by I_n and O the $n\times n$ identity matrix and the zero matrix, respectively. The notations $\oplus$ and $\otimes$ denote the direct sum and the Kronecker product, respectively.

The multivariate linear model can be described as follows:

$$Y = XK + E, \tag{1}$$

where K is an unknown $q\times p$ matrix of coefficient parameters, E is an $n\times p$ matrix of random errors, Y is an $n\times p$ matrix of n observations on p characteristics, and X is a known $n\times q$ model (design) matrix whose all elements in the first column are 1. We assume that $\mathrm{Var}(Y)=\mathrm{Var}(E)=W$ is an unknown positive definite matrix, $E\sim MN(O,W)$ and $Y\sim MN(XK,W)$, where the notation $MN(\mu,\Xi)$ denotes a matrix normal distribution with mean μ and covariance matrix Ξ. Let Y_i be the ith column of Y' and let $\mathrm{Var}(Y_i)=\Sigma$ is a positive definite matrix. If we assume that the usual i.i.d. covariance structure, then

$$W = I_n\otimes\Sigma. \tag{2}$$

Consider the test of the linear hypothesis

$$\mathrm{H}_0\colon LK = O \quad \text{vs.} \quad H_\alpha\colon LK\neq O, \tag{3}$$

where L is an $s\times q$ constraint matrix of rank s such that LK is estimable. Let $P = X(X'X)^-X'$ and $P_0 = [X(X'X)^-L'][L(X'X)^-L']^{-1}[L(X'X)^-X']$. Assuming the i.i.d. covariance structure (2), we have that $Q = Y'(I_n-P)Y\sim W_p(n-r,\Sigma,O)$ and $R = Y'P_0Y\sim W_p(s,\Sigma,\Delta)$, where $r=\mathrm{rank}\,X$, and $W_p(s,\Sigma,\Delta)$ denotes a noncentral Wishart distribution with s degrees of freedom, parameter matrix Σ and noncentrality parameter $\Delta=(LK)'[L(X'X)^-L']^{-1}(LK)$. Under the matrix-normal i.i.d. model defined in (1) and (2), common statistics for testing hypothesis (3) are functions of Q and R. These include, for example, the Lawley–Hotelling trace statistic, $\mathrm{LH}=(n-r)\mathrm{tr}[(QR)^{-1}]$; Wilks' lambda statistic, $\Lambda=|Q|/|Q+R|$; and Pillai's trace statistic, $\mathrm{PT}=\mathrm{tr}[R(Q+R)^{-1}]$, among others. We shall denote an arbitrary member of this group of test statistics by $f(Q,R)$.

Young et al. [10, Theorem 1] investigated the following theorem.

Theorem 1.1. *Given matrices $A,B,C,D\in\mathbb{C}^{p\times n}$. The matrix equations*

$$AXA^* = BB^* \tag{4}$$

and

$$CXC^* = DD^* \tag{5}$$

have a common Hermitian nonnegative-definite solution if and only if

$$AA^+B = B \tag{6}$$

and

$$[C(I_n - A^+A)][C(I_n - A^+A)]^+(D - CA^+B) = (D - CA^+B). \tag{7}$$

If a common Hermitian nonnegative-definite solution exists, then a representation of the general common Hermitian nonnegative-definite solution is $X = UU^*$ *with*

$$U = A^+B + (I_n - A^+A)[C(I_n - A^+A)]^+(D - CA^+B)$$
$$+ (I_n - A^+A)\{I_n - [C(I_n - A^+A)]^+[C(I_n - A^+A)]\}Z,$$

where Z *is free to vary over* $\mathbb{C}^{n\times n}$.

Furthermore, Young et al. used Theorem 1.1 to solve the following statistical problem [10, Theorems 2, 3] which is concerned in testing linear hypotheses about regression coefficients in the multivariate linear model.

Problem 1.1. *For a multivariate linear model* (1) *with* $Y \sim MN(XK, V \otimes \Sigma)$, *determine necessary and sufficient conditions on the matrix* $V \in \mathbb{R}_n^>$ *such that the statistics for the hypothesis testing* (3) *have the same distribution as under the i.i.d. covariance structure* (2).

But Groß [5] has given a counterexample for pointing out that Theorem 1.1 is not true, thus the solution of Problem 1.1 obtained by Young et al. is also incorrect. However, up to now, nobody (we knew) has obtained a corrective representation of the general common Hermitian nonnegative-definite solution to the matrix equations (4) and (5).

The aim of this paper is to correct Theorem 1.1 (see Section 2 below) and to solve Problem 1.1 (see Section 3 below). Moreover, this paper can also be viewed as a supplementary version of [1,3–6,11].

2. A revision of Theorem 1.1

In this section, we correct Theorem 1.1. Let $\mathbb{U}_n$ be the subset of $\mathbb{C}^{n\times n}$ consisting of all unitary matrices. We first introduce the following three lemmas.

Lemma 2.1 (see Rao and Mitra [9, p. 17] or Young et al. [10, Lemma 2]). *Given matrices* $F, G \in \mathbb{C}^{m\times n}$. *Then* $FF^* = GG^*$ *if and only if* $G = FT$ *for some* $T \in \mathbb{U}_n$.

Lemma 2.2 (see Buxton et al. [2, p. 270]). *Given matrices* $M \in \mathbb{C}^{m\times p}$ *and* $N \in \mathbb{C}^{m\times n}$. *Then the matrix equation* $MX = N$ *has a solution if and only if* $MM^+N = N$. *If a solution exists, then a representation of the general solution is* $X = M^+N + (I - M^+M)Z$, *where* Z *is free to vary over* $\mathbb{C}^{p\times n}$.

Lemma 2.3 (see Baksalary [1, Theorem 1] or Groß [4, p. 124]). *Given matrices $A, B \in \mathbb{C}^{m\times n}$. Then the matrix equation* (4) *has a Hermitian nonnegative-definite solution if and only if* (6) *is satisfied. If a Hermitian nonnegative-definite solution exists, then a representation of the general Hermitian nonnegative-definite solution is*

$$X = [A^+B + (I_n - A^+A)Y][A^+B + (I_n - A^+A)Y]^*, \tag{8}$$

where Y is free to vary over $\mathbb{C}^{n\times n}$.

To correct Theorem 1.1, the following two lemmas are also required.

Lemma 2.4. *Given $A, B \in \mathbb{C}^{m\times n}$ and $C, D \in \mathbb{C}^{p\times n}$ satisfying* (6). *Then the following* (a) *and* (b) *hold.*

(a) *The matrix equation* (5) *has a solution in the form of* (8) *if and only if there exists $T \in \mathbb{U}_n$ such that*

$$[C(I_n - A^+A)][C(I_n - A^+A)]^+(DT - CA^+B) = DT - CA^+B; \tag{9}$$

(b) *If* (9) *holds for some $T \in \mathbb{U}_n$, then X defined in* (8) *is a solution to the matrix equation* (5) *if and only if*

$$\begin{aligned} Y = {} & [C(I_n - A^+A)]^+(DT - CA^+B) + Z \\ & - [C(I_n - A^+A)]^+C(I_n - A^+A)Z, \end{aligned} \tag{10}$$

where Z is free to vary over $\mathbb{C}^{n\times n}$ and $T \in \mathbb{U}_n$ is a parameter matrix satisfying (9).

Proof. (a) *The "only if" part*. Suppose the matrix equation (5) has a solution in the form of (8). Then there exists $Y_0 \in \mathbb{C}^{n\times n}$ such that

$$[CA^+B + C(I_n - A^+A)Y_0][CA^+B + C(I_n - A^+A)Y_0]^* = DD^*. \tag{11}$$

Using Lemma 2.1 deduces that there exists $T \in \mathbb{U}_n$ such that

$$CA^+B + C(I_n - A^+A)Y_0 = DT, \tag{12}$$

i.e., Y_0 is a solution to the matrix equation $C(I_n - A^+A)Y = DT - CA^+B$ for some $T \in \mathbb{U}_n$. This, together with Lemma 2.2, implies that (9) holds for some $T \in \mathbb{U}_n$.

The "if" part. Suppose (9) holds for some $T \in \mathbb{U}_n$. Then, by Lemma 2.2, the matrix $Y_0 = [C(I_n - A^+A)]^+(DT - CA^+B)$ is a solution to the matrix equation $C(I_n - A^+A)Y = DT - CA^+B$. That is, (12) holds, and hence (11) holds. This implies that the matrix equation (5) have a solution in the form of (8).

(b) Since (9) holds for some $T \in \mathbb{U}_n$, it follows from (a) that the matrix equation (5) has at least a solution in the form of (8). Furthermore, X defined in (8) is a solution to the matrix equation (5) if and only if

$$[CA^+B + C(I_n - A^+A)Y][CA^+B + C(I_n - A^+A)Y]^* = DD^*. \tag{13}$$

Using Lemma 2.1, we have that (13) holds if and only if $CA^+B + C(I_n - A^+A)Y = DT$ for some $T \in \mathbb{U}_n$. This, together with Lemma 2.2, implies that X defined in (8) is a solution to the matrix equation (5) if and only if Y has the form (10).

The proof is completed. □

Lemma 2.5. *Given* $A, B \in \mathbb{C}^{m\times n}$ *and* $C, D \in \mathbb{C}^{p\times n}$. *Furthermore, denote*

$$\begin{cases} \hat{F} = \{I_p - [C(I_n - A^+A)][C(I_n - A^+A)]^+\}D, \\ \hat{G} = \{I_p - [C(I_n - A^+A)][C(I_n - A^+A)]^+\}CA^+B. \end{cases} \tag{14}$$

Then there exists $T \in \mathbb{U}_n$ *satisfying* (9) *if and only if*

$$\hat{F}\hat{F}^* = \hat{G}\hat{G}^*. \tag{15}$$

When (15) *is met, the set* Γ *of all* $T \in \mathbb{U}_n$ *satisfying* (9) *can be obtained by using the following steps*:

Step 1: *Find* $\hat{P} \in \mathbb{U}_p$, $\hat{Q} \in \mathbb{U}_n$ *and diagonal* $\hat{\Sigma} \in \mathbb{R}_s^{\geqslant}$ $(s = \operatorname{rank} \hat{F})$ *satisfying*

$$\hat{F} = \hat{P}(\hat{\Sigma} \oplus O)\hat{Q}; \tag{16}$$

Step 2: *Calculate* $\hat{G}_1 \in \mathbb{C}^{s\times n}$ *and* $\hat{G}_2 \in \mathbb{C}^{(p-s)\times n}$ *according to*

$$\begin{bmatrix} \hat{G}_1 \\ \hat{G}_2 \end{bmatrix} = (\hat{\Sigma}^{-1} \oplus I_{p-s})\hat{P}^*\hat{G}\hat{Q}^*; \tag{17}$$

Step 3: *Find a matrix* $S \in \mathbb{C}^{(n-s)\times n}$ *satisfying* $\begin{bmatrix} \hat{G}_1 \\ S \end{bmatrix} \in \mathbb{U}_n$;

Step 4: *Construct the set* Γ *by*

$$\Gamma = \left\{ \hat{Q}^* \begin{bmatrix} \hat{G}_1 \\ US \end{bmatrix} \hat{Q} \,\middle|\, U \in \mathbb{U}_{n-s} \right\}. \tag{18}$$

Proof. It follows from (14) that (9) can be written as

$$\hat{F}T = \hat{G}. \tag{19}$$

By Lemma 2.1, there exists $T \in \mathbb{U}_n$ satisfying (19) if and only if (15) holds.

When (15) is met, we have from (16) and (17) that

$$\hat{G}_2 = O \tag{20}$$

and

$$\hat{G}_1\hat{G}_1^* = I_s. \tag{21}$$

Substituting (20) into (17) yields

$$\hat{G} = \hat{P}\begin{bmatrix} \hat{\Sigma}\hat{G}_1 \\ O \end{bmatrix}\hat{Q}. \tag{22}$$

Combining (16) and (22) gives that (9) (or equivalently, (19)) is equivalent to $[I_s \quad O]\hat{Q}T = \hat{G}_1\hat{Q}$. This, together with (21), implies that the set Γ of all $T \in \mathbb{U}_n$

satisfying (9) is given by

$$\Gamma = \left\{ \hat{Q}^* \begin{bmatrix} \hat{G}_1 \\ U_0 \end{bmatrix} \hat{Q} \,\middle|\, U_0 \in \mathbb{C}^{(n-s)\times n} \text{ satisfying } \begin{bmatrix} \hat{G}_1 \\ U_0 \end{bmatrix} \in \mathbb{U}_n \right\}. \tag{23}$$

It is easy to see from Step 3 that (23) and (18) are equivalent.

The proof is completed. □

Remark 2.1. If $s = n$, then the matrix S in Step 3 of Lemma 2.5 vanishes, and hence (18) turns into $\Gamma = \{\hat{Q}^* \hat{G}_1 \hat{Q}\}$. If $s = 0$, then the matrix $\hat{G}_1$ in (17) vanishes, and thus (18) turns into $\Gamma = \mathbb{U}_n$. Therefore, for these two special cases, Steps 3 and 4 in Lemma 2.5 can be simplified.

Example 2.1. Consider the matrix equations (4) and (5) with the parameter matrices

$$A = C = D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Obviously, $m = n = p = 2$. According to (14), we have $\hat{F} = D$ and $\hat{G} = B$. It is easily verified that (15) holds. By Lemma 2.5, the set Γ of all $T \in \mathbb{U}_n$ satisfying (9) is

$$\Gamma = \left\{ \begin{bmatrix} 0 & 1 \\ u & 0 \end{bmatrix} \,\middle|\, u \in \mathbb{U}_1 \right\} = \left\{ \begin{bmatrix} 0 & 1 \\ u & 0 \end{bmatrix} \,\middle|\, u \in \mathbb{C},\ |u| = 1 \right\}.$$

Combining Lemmas 2.3, 2.4 and 2.5, Theorem 1.1 can be corrected as follows.

Theorem 2.1. *Given $A, B \in \mathbb{C}^{m\times n}$ and $C, D \in \mathbb{C}^{p\times n}$, and further let $\hat{F}$, $\hat{G}$ and Γ be as previously defined. Then the matrix equations* (4) and (5) *have a common Hermitian nonnegative-definite solution if and only if* (6) *and* (15) *hold. If a common Hermitian nonnegative-definite solution exists, then a representation of the general common Hermitian nonnegative-definite solution is given by* (8) with (10), *where Z and T are free to vary over $\mathbb{C}^{n\times n}$ and Γ, respectively.*

The following example and remark show the difference between Theorem 1.1 and Theorem 2.1.

Example 2.2. Consider the matrix equations (4) and (5) in Example 2.1. It is easily verified that (6) and (15) hold. By Theorem 2.1, the matrix equations (4) and (5) have at least a common Hermitian nonnegative-definite solution, and further a representation of the general common Hermitian nonnegative-definite solution is $X = \begin{bmatrix} 1 & y_2^* \\ y_2 & y_1 y_1^* + y_2 y_2^* \end{bmatrix}$, where $y_1, y_2 \in \mathbb{C}$ are two any parameters.

Remark 2.2. For the matrix equations (4) and (5) in Example 2.1, it is easily verified that (7) does not holds. By Theorem 1.1, there does not exist any common Hermitian nonnegative-definite solution to the matrix equations (4) and (5), which contradicts that $X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is a common Hermitian nonnegative-definite solution to the matrix equations (4) and (5) (see Example 2.2). Therefore, the condition '(6) and (7) hold' given in Theorem 1.1 is not a necessary and sufficient condition for solvability. However, following the argument in the proof of Theorem 1.1 (see [10]), we have that the condition '(6) and (7) hold' is a sufficient condition for solvability.

3. The solution to Problem 1.1

This section solves Problem 1.1 proposed in Section 1. We first show the following corollary which is a special case of Theorem 2.1.

Corollary 3.1. *Let $G_1, G_2 \in \mathbb{R}_n^{\geqslant}$ be idempotent matrices such that $G_1 G_2 = O$, and further assume $G_0 = I_n - G_1 - G_2$. Then the matrix equations*

$$G_i X G_i = G_i, \quad i = 1, 2 \tag{24}$$

have at least a common symmetric nonnegative-definite solution, and further a representation of the general common symmetric nonnegative-definite solution is

$$X = [G_1 + G_2 T + G_0 Z][G_1 + G_2 T + G_0 Z]', \tag{25}$$

where Z and T are free to vary over $\mathbb{R}^{n \times n}$ and $\mathbb{O}_n$, respectively.

To solve Problem 1.1, we need the following theorem concerning the common symmetric positive-definite solutions to the matrix equations (24).

Theorem 3.1. *Let G_1, G_2 and G_0 are as previously defined. Then X in the form of* (25) *is a common symmetric positive-definite* (*i.e., nonsingular*) *solution to the matrix equations* (24) *if and only if*

$$\mathcal{R}(G_1 + T' G_2) \cap \mathcal{R}(Z' G_0) = \{O\}, \tag{26}$$

$$\mathcal{N}(Z') \cap \mathcal{N}(G_1 + G_2) = \{O\} \tag{27}$$

and

$$G_1 T' G_2 = O \tag{28}$$

are simultaneously satisfied.

Proof. Firstly, it follows from (25) and the definitions of G_1, G_2 and G_0 that

$$\begin{aligned}\operatorname{rank} X &= \operatorname{rank}(G_1 + G_2T + G_0Z)\\ &\leqslant \operatorname{rank}\,(G_1 + G_2T) + \operatorname{rank}\,(G_0Z)\\ &\leqslant \operatorname{rank}\,(G_1 + G_2T) + \operatorname{rank}\, G_0\\ &\leqslant \operatorname{rank}\, G_1 + \operatorname{rank}\,(G_2T) + \operatorname{rank}\, G_0\\ &= \operatorname{rank}\, G_1 + \operatorname{rank}\, G_2 + \operatorname{rank}\, G_0\\ &= n.\end{aligned}$$

Therefore, $\operatorname{rank} X = n$ if and only if

$$\operatorname{rank}\,(G_1 + G_2T + G_0Z) = \operatorname{rank}(G_1 + G_2T) + \operatorname{rank}(G_0Z), \tag{29}$$

$$\operatorname{rank}(G_0Z) = \operatorname{rank}\, G_0 \tag{30}$$

and

$$\operatorname{rank}(G_1 + G_2T) = \operatorname{rank}\, G_1 + \operatorname{rank}(G_2T) \tag{31}$$

are simultaneously satisfied.

Secondly, we deduce from [7] that (29) holds if and only if (26) and

$$\mathcal{R}(G_1 + G_2T) \cap \mathcal{R}(G_0Z) = \{O\} \tag{32}$$

are simultaneously satisfied. While (32) is inherently satisfied. Indeed, if $y \in \mathcal{R}(G_1 + G_2T) \cap \mathcal{R}(G_0Z)$, then

$$y = (G_1 + G_2T)a \tag{33}$$

and

$$y = G_0Zb \tag{34}$$

for some vectors a and b. Using (34) and the definitions of G_1, G_2 and G_0, we have that $G_1y = O$ and $G_2y = O$. This, together with (33), gives $y = O$. Therefore, (32) holds.

Thirdly, it follows from [8, Theorem 3.4.17] that (30) is equivalent to $\mathcal{N}(Z') \cap \mathcal{R}(G_0) = \{O\}$. This, together with $\mathcal{R}(G_0) = \mathcal{N}(G_1 + G_2)$, implies that (30) is equivalent to (27).

Fourthly, by [7], we have (31) holds if and only if

$$\mathcal{R}(G_1) \cap \mathcal{R}(G_2T) = \{O\}, \quad \mathcal{R}(G_1) \cap \mathcal{R}(T'G_2) = \{O\} \tag{35}$$

are satisfied. Sine $G_1 \in \mathbb{R}_n^{\geqslant}$ is idempotent, there exists $\tilde{Q} \in \mathbb{O}_n$ such that $G_1 = \tilde{Q}(I_k \oplus O)\tilde{Q}'$, which implies $\mathcal{R}(G_1) \oplus \mathcal{N}(G_1) = \mathbb{R}^{n\times 1}$. Thus, (35) is equivalent to $\mathcal{R}(G_2T) \subseteq \mathcal{N}(G_1)$ and $\mathcal{R}(T'G_2) \subseteq \mathcal{N}(G_1)$. Note that $\mathcal{R}(G_2T) \subseteq \mathcal{N}(G_1)$ is inherently satisfied since $G_1G_2 = O$. While $\mathcal{R}(T'G_2) \subseteq \mathcal{N}(G_1)$ is equivalent to $G_1T'G_2y = O$ for any $y \in \mathbb{R}^{n\times 1}$ (i.e., (28) holds).

Combining the above four aspects completes the proof. □

Remark 3.1. By the definitions of G_1, G_2 and G_0, it is easily verified that $T = I_n$ and $Z = G_0$ satisfy (26)–(28). That is, the matrix equations (24) have at least a common symmetric positive-definite solution.

Based on Theorem 3.1 and [10, Lemma 1], the solution to Problem 1.1 can be stated (i.e., [10, Theorems 2 and 3] can be corrected) as follows.

Theorem 3.2. *Let $Y \sim MN(\mu, V \otimes \Sigma)$ with $V \in \mathbb{R}_n^>$ and $\Sigma \in \mathbb{R}_p^>$, and G_1, G_2 and G_0 are as previously defined. Then, $Y'G_iY \sim W_p(k_i, \Sigma, \Delta_i)$, $i = 1, 2$, are independent with $k_i = \operatorname{rank} G_i$ and $\Delta_i = \mu' G_i \mu$, $i = 1, 2$, if and only if $V = [G_1 + G_2T + G_0Z][G_1 + G_2T + G_0Z]'$, where $Z \in \mathbb{R}^{n\times n}$ and $T \in \mathbb{O}_n$ are two parameter matrices satisfying (26)–(28).*

Theorem 3.3. *Let $Y \sim MN(XK, V \otimes \Sigma)$ with $V \in \mathbb{R}_n^>$ and $\Sigma \in \mathbb{R}_p^>$, and P and P_0 be as previously defined. Then the distribution of the test statistic $f(Q, R)$ is identical to the distribution of $f(Q, R)$ assuming the model $Y \sim MN(XK, I_n \otimes \Sigma)$ if and only if*

$$V = [I_n - P + P_0T + (P - P_0)Z][I_n - P + P_0T + (P - P_0)Z]',$$

where $Z \in \mathbb{R}^{n\times n}$ and $T \in \mathbb{O}_n$ are two parameter matrices satisfying

$$\begin{cases} \mathcal{R}(I_n - P + T'P_0) \cap \mathcal{R}(Z'(P - P_0)) = \{O\}, \\ \mathcal{N}(Z') \cap \mathcal{N}(I_n - P + P_0) = \{O\}, \\ (I_n - P)T'P_0 = O. \end{cases} \tag{36}$$

Example 3.1. Consider the multivariate linear model (1) and the test (3) with the parameter matrices [10]:

$$\Sigma = \begin{bmatrix} 4 & 0.1 \\ 0.1 & 3 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 2 & 9 \\ 1 & 5 & 13 \\ 1 & 5 & 7 \\ 1 & 10 & 17 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Choosing

$$T = I_4 \in \mathbb{O}_n, \quad Z = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix} \in \mathbb{R}^{n\times n},$$

we have that (36) is satisfied. By Theorem 3.3, one possible independence distribution-preserving covariance structure for test (3) is

$$W = V \otimes \Sigma = \begin{bmatrix} 7 & 0.175 & 3 & 0.075 & 3 & 0.075 & 3 & 0.075 \\ 0.175 & 5.25 & 0.075 & 2.25 & 0.075 & 2.25 & 0.075 & 2.25 \\ 3 & 0.075 & 7 & 0.175 & 3 & 0.075 & 3 & 0.075 \\ 0.075 & 2.25 & 0.175 & 5.25 & 0.075 & 2.25 & 0.075 & 2.25 \\ 3 & 0.075 & 3 & 0.075 & 7 & 0.175 & 3 & 0.075 \\ 0.075 & 2.25 & 0.075 & 2.25 & 0.175 & 5.25 & 0.075 & 2.25 \\ 3 & 0.075 & 3 & 0.075 & 3 & 0.075 & 7 & 0.175 \\ 0.075 & 2.25 & 0.075 & 2.25 & 0.075 & 2.25 & 0.175 & 5.25 \end{bmatrix}.$$

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